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# Two-Particle Collisions. II. Coulomb Collisions in the Laboratory System of Coordinates

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This paper presents a classical theory of Coulomb collisions, as a particular case of the previously published general theory of two-particle collisions. There are derived and presented explicitly, in the laboratory system of coordinates, both the energy and angular relations for two-particle collisions and the cross sections related to them. Apart from the exact formulas, approximations are obtained, which should prove to be both sufficiently exact for the majority of calculations and at the same time fairly simple, while emphasizing the main features of the Coulomb interaction.

## I. INTRODUCTION

OF all the interactions occurring in nature, a special role is played by those whose forces decrease with the square of the distance, i.e., by the Coulomb and gravitational forces.

Clear understanding of the interaction (collision) processes of many-particle systems invariably requires a profound knowledge of the two-particle collision process. As in other cases, so too for the Coulomb forces, full analysis has not been made of the collision process in the laboratory system of coordinates, the significance of which has been emphasized for the first time by Chandrasekhar.[1] The analysis of this problem up to now has been highly fragmentary and incomplete, and it has concerned rather particular cases only. It is enough to mention here the well-known Rutherford formula concerning the scattering of charged particles, which has played a decisive role in the knowledge of the structure of the atom, or the Bohr interpretation of some atomic-collision phenomena. These problems, like a number of others (Thompson's theory of ionization, for example), concern the scattering on particles at rest; the approximation resulting therefrom is sufficiently correct only within a certain range of the phenomena under consideration, and the failure to realize this has in many instances led to completely erroneous conclusions.

A close study of the problem of two-particle collision, already partly carried out by the author,[2,3] yielded an extremely simple explanation for a number of phenomena in the field of atomic collisions. The aim of the present paper has been to analyze the Coulomb interactions more completely on the basis of the general theory of collisions given by the author previously.[4]

## II. BASIC CROSS SECTIONS FOR COULOMB FORCES

The relations derived in Paper I as a result of general considerations, arising only from the laws of conservation of energy and momentum, are easily applied to forces decreasing with the square of the distance, i.e., to the gravitational and Coulomb forces. As we have already shown, the characteristic features of the interactions are expressed by the scattering angle in the center-of-mass system. In our formalism, therefore, the dependence on the interaction law enters the basic cross sections as well as the derivative quantities through the function $F'(\Psi_g,\theta)$ defined by the Eq. (I.64), which in this case takes the very simple form $F'(k/\mu V^2)^2$. The constant $k$ in the above expression determines the coupling force between the interacting bodies. For the gravitational field, the constant $k$ is $m_1m_2G$, where $m_1$ and $m_2$ are the masses of the colliding particles, and $G$ is the gravitational constant ($G=6.7\times10^{-8}$ g$^{-1}$ cm$^3$ sec$^{-1}$); in the case of the Coulomb field we have: $k=q_1q_2$, where $q_1$ and $q_2$ are the charges of the particles under consideration. Since the function $F'$ enters directly as a factor in all the cross sections describing the interaction of particles, the form of all relations is independent of the constant $k$.

Taking this into consideration, we shall subsequently speak only of the Coulomb interaction, while bearing in mind that by substituting $G=6.7\times10^{-8}$ g$^{-1}$ cm$^3$ sec$^{-1}$ we obtain relations for gravitating particles.

The above being considered, the fundamental cross sections as given by (I.68), (I.72), and (I.79) will take the form

$$\sigma_{\Delta E}=\frac{\sigma_0}{E_1E_2}\frac{1}{K_{12}}\frac{1}{\pi}\times\int\int f_V\frac{d(1/\cos^2\Psi_g)}{(W_{\Psi_g}(1/\cos^2\Psi_g,\Delta E,\theta))^{1/2}}f(\theta)d\theta\,,\quad(1)$$

$$\sigma_{\Delta E,\cos\vartheta}=\frac{\sigma_0}{E_2{}^2}\frac{(1+\Delta E/E_2)^{1/2}}{\xi^{5/2}}\frac{1}{2\sqrt{2}m_2v_1v_2}\frac{1}{\pi}\times\int\frac{f(\theta)d\theta}{(W_\xi(\Delta E,\cos\vartheta,\theta))^{1/2}}\,,\quad(2)$$

[1] S. Chandrasekhar, Astrophys. J. **93**, 285 (1941).
[2] M. Gryziński, Phys. Rev. **107**, 1471 (1957).
[3] M. Gryziński, Phys. Rev. **115**, 347 (1959).
[4] M. Gryziński, preceding paper, Phys. Rev. **138**, A305 (1965), hereafter cited as I.

$$\sigma_{\Delta E,\cos\vartheta,\phi}=\frac{\sigma_0}{E_2^2}\frac{(1+\Delta E/E_2)^{1/2}}{\xi^{5/2}}\frac{1}{2\sqrt{2}m_2v_1v_2}\frac{1}{\pi}\times\int\frac{f[\theta,\varphi(\phi)]}{(W_\xi(\Delta E,\cos\vartheta,\theta))^{1/2}}d\theta,\quad(3)$$

where we have introduced the notation

$$\sigma_0=\pi(q_1q_2)^2,\quad(4)$$

$$f_V=\left(\frac{v_1}{v_2}\right)^2\left(\frac{v_2^2}{v_1^2+v_2^2-2v_1v_2\cos\theta}\right)^{3/2},\quad(5)$$

while the remaining symbols are the same as those introduced in Paper I. First, we shall proceed to calculate the above integrals under various assumptions with respect to the distribution function $f(\theta)$; next, we shall give expressions for several most important derivative quantities; and finally, we shall prove the correspondence of the formulas obtained to a number of already known relations.

## III. CALCULATION OF ENERGY-EXCHANGE CROSS SECTIONS

Performing the integration in Eq. (1) over $\Psi_g$ in the range $W_{\Psi_g}=(1/\cos^2\Psi_g,\Delta E,\theta)\geq 0$, we obtain the cross section for a collision in which the scattered particle undergoes a change in energy $\Delta E$:

$$\sigma_{\Delta E}=\frac{\sigma_0}{\Delta E^2}\frac{\operatorname{sgn}(-\Delta E)}{E_1}\int_{\theta_1(\Delta E)}^{\theta_2(\Delta E)}f_V\left[1-\frac{E_1}{E_2}-\frac{v_1}{v_2}\left(1-\frac{m_1}{m_2}\right)\cos\theta-2\frac{E_1}{\Delta E}\sin^2\theta\right]f(\theta)d\theta,\quad(6)$$

where the integration limits for an angle $\theta$ are determined by the condition (I.93).

Having obtained the final form of the cross section $\sigma_{\Delta E}$, it will be rather easy to give explicitly certain derivative relations which, however, have direct counterparts in the physical phenomena observed. Starting from (6), we can immediately give the cross sections for:

(i) a collision in which the particle loses energy in the interval $U\leq|\Delta E|\leq|\Delta E^-_{\max}|$ (which corresponds to the ionization process in atomic collisions),

$$Q=\int_{-U}^{-|\Delta E^-_{\max}|}\sigma_{\Delta E}d(\Delta E)=\frac{\sigma_0}{UE_1}\int_{\theta_1(-U)}^{\theta_2(-U)}f_V\left\{x\left[x\frac{E_1}{U}\sin^2\theta+1-\frac{E_1}{E_2}-\frac{v_1}{v_2}\left(1-\frac{m_1}{m_2}\right)\cos\theta\right]\right\}\Bigg|_{x=U/|\Delta E^-(\theta)|}^{x=1}f(\theta)d\theta;\quad(7)$$

(ii) stopping power of a quantized particle with a minimum excitation energy $U$,

$$S=\int_{-U}^{-|\Delta E^-_{\max}|}\sigma_{\Delta E}\Delta Ed(\Delta E)$$

$$=\frac{\sigma_0}{E_1}\int_{\theta_1(-U)}^{\theta_2(-U)}f_V\left\{\left[1-\frac{E_1}{E_2}-\frac{v_1}{v_2}\left(1-\frac{m_1}{m_2}\right)\cos\theta\right]\ln\frac{|\Delta E^-(\theta)|}{U}+2\frac{E_1}{U}\sin^2\theta\left(1-\frac{U}{|\Delta E^-(\theta)|}\right)\right\}f(\theta)d\theta,\quad(8)$$

where $\Delta E^-(\theta)$ is determined by (I.83) while $\theta(-U)$, as in the case (4), is similarly determined by (I.93), in which we have inserted $\Delta E=-U$.

Now, the relations (I.123) and (I.83) determine the stopping power of the free particle:

$$S_D=\int_{\Delta E^-_{\max}}^{\Delta E^+_{\max}}\sigma_{\Delta E}\Delta Ed\Delta E=\frac{\sigma_0}{E_1}\int_0^\pi f_V\left[1-\frac{E_1}{E_2}-\frac{v_1}{v_2}\left(1-\frac{m_1}{m_2}\right)\cos\theta\right]\ln\left[1+\left(\frac{\mu V^2D}{q_1q_2}\right)^2\right]f(\theta)d\theta,\quad(9)$$

which, as we well know, is divergent with respect to the impact parameter $D$. The last formula is identical with the dependence for the gravitating bodies obtained by Chandrasekhar.[5]

In the four expressions derived above, essential for the interpretation of the majority of phenomena taking place in collisions of charged particles, there appears a function $f_V$, characteristic of that interaction, which determines the nature of the relations obtained. The graph of this function is given in Fig. 1.

[5] S. Chandrasekhar, Astrophys. J. **93**, 323 (1941).

In two limiting cases, $v_2\gg v_1$ and $v_2\ll v_1$, $f_V$ leads to a dependence very common in atomic collisions:

$$f_V\to(v_1/v_2)^2\sim 1/E_2\quad\text{for}\quad v_2\gg v_1$$
$$\to v_1/v_2\sim\sqrt{E_2}\quad\text{for}\quad v_2\ll v_1.$$

In the region of large energies of the test particle (in Coulomb collisions, a particle of large energy is one whose velocity is much greater than that of the particle with which it collides), the effects of interaction decrease inversely as the particle energy; whereas, in the region of low energies, the effects of interaction decrease proportionally to the square root of the particle energy.

For $\mathbf{v}_2 \to \mathbf{v}_1$, there appears a strong resonance between the test particle and the field particles, which, *inter alia*, is directly connected with the appearance of oscillations in the plasma. Another characteristic feature of the Coulomb interaction is the decrease of the cross section for energy exchange with the square of $\Delta E$:

$$\sigma_{\Delta E} \propto 1/(\Delta E)^2 .$$

In conclusion, we can state that the inverse proportionality of the cross section $\sigma_{\Delta E}$ to $\Delta E$, as well as the form of the function $f_V$, immediately gives a qualitative interpretation of the majority of phenomena occurring in Coulomb collisions.

Inserting $v_1=0$ in the above formulas (scattering on the field particles at rest), we obtain the well-known relations:

$$\sigma_{\Delta E}=\frac{2\pi(q_1q_2)^2}{m_1v_2^2}\frac{1}{\Delta E^2}$$

$$\text{with the condition } K_{12}E_2 \leq \Delta E \leq 0, \quad (10)$$

$$Q=\frac{2\pi(q_1q_2)^2}{m_1v_2^2}\frac{1}{U}\left(1-\frac{U}{K_{12}E_2}\right), \quad (11)$$

$$S=\frac{2\pi(q_1q_2)^2}{m_1v_2^2}\ln\frac{K_{12}E_2}{U}, \quad (12)$$

$$S_D=\frac{2\pi(q_1q_2)^2}{m_1v_2^2}\ln\left[1+\left(\frac{\mu v_2^2 D}{q_1q_2}\right)^2\right], \quad (13)$$

of which (11) is identical with the dependence derived by Thompson,[6] whereas (13) is an equivalent of the Bethe formula[7] provided we take into account that the maximum impact parameter $D$ at which a field particle can gain an energy amount greater than $U$ is determined by

$$q_1q_2/D \geq U .$$

Here it is necessary to draw attention to the fact that the approximation $v_1 \to 0$ is not equivalent to the approximation $v_2 \gg v_1$, and that it differs much from the latter if we consider a collision with a very small energy exchange ($\Delta E \ll E_1$):

$$\sigma_{\Delta E} \propto 1/\Delta E^2 \quad \text{for} \quad v_1=0$$
$$\propto 1/\Delta E^3 \quad \text{for} \quad \Delta E \ll E_1; \text{ however, } v_2 \gg v_1 .$$

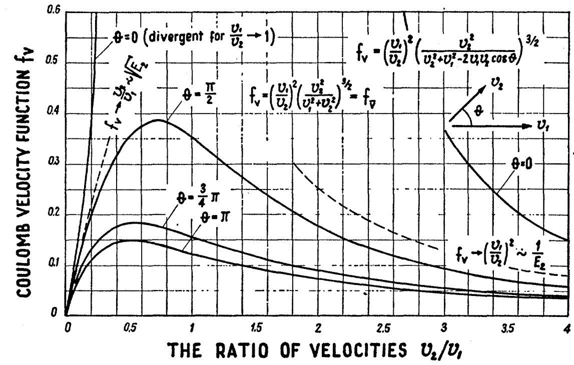


FIG. 1. Coulomb velocity function $f_V$ which determines the basic dependence of energy-exchange processes on the velocity of the colliding particles.

## IV. ANALYSIS OF ENERGY CROSS SECTIONS IN THE CASE OF ISOTROPIC DISTRIBUTION OF FIELD PARTICLES

In order to give explicit expressions for $\sigma_{\Delta E}$, $Q$, $S$, and $S_D$, it is necessary to define the velocity distribution of the field particles. In the most common case of isotropic velocity distribution ($f(\theta)=\frac{1}{2}\sin\theta$), the analytical form for the arbitrary values of $m_1$ and $m_2$ can be derived only for the expressions $\sigma_{\Delta E}$ and $S_D$, which has already been done[2,3]; nevertheless, to give a full picture of the case, we shall quote once more the results of our final calculations, especially in view of the fact that some errors have crept into our previous publication.

(i) Cross section for a collision in which the test particle undergoes a change in energy $\Delta E$:

$$\sigma_{\Delta E}=(\sigma_0/\Delta E^2)\,(\mathrm{sgn}(-\Delta E)/E_1)g^{\mathrm{ex}}{}_\sigma , \quad (14)$$

$$g_\sigma{}^{\mathrm{ex}}=\frac{1}{2}\left(\frac{v_1}{v_2}\right)^2\left(1+\left(\frac{v_2}{v_1}\right)^2-2\frac{v_2}{v_1}x\right)^{-1/2}\left\{-\frac{2}{3}\frac{E_1}{\Delta E}\left(x-\frac{v_2}{v_1}\right)^2+\left(x-\frac{v_2}{v_1}\right)\left[1-\frac{m_1}{m_2}-\frac{4}{3}\frac{E_1}{\Delta E}\left(1+\frac{v_2^2}{v_1^2}\right)\right]\right.$$
$$\left.+\left[\frac{m_1}{m_2}\frac{v_2^2}{v_1^2}-1-2\frac{E_1}{\Delta E}\frac{v_2^2}{v_1^2}+\frac{4}{3}\frac{E_1}{\Delta E}\left(1+\frac{v_2^2}{v_1^2}\right)^2\right]\right\}\Bigg|_{x=\cos\theta_2(\Delta E)}^{x=\cos\theta_1(\Delta E)}, \quad (15)$$

where $\cos\theta_{1,2}(\Delta E)$ is given by Eq. (I.93). If we consider a collision for which the energy change is in the range

$$|\Delta E| \leq \Delta E_{\theta=0}=K_{12}E_2\left(1+\frac{m_1v_1}{m_2v_2}\right)\left(1-\frac{v_1}{v_2}\right) \quad \text{or} \quad |\Delta E| \leq \Delta E_{\theta=\pi}=K_{12}E_2\left(1+\frac{v_1}{v_2}\right)\left(1-\frac{m_1v_1}{m_2v_2}\right), \quad (16)$$

[6] J. J. Thompson, Phil. Mag. **23**, 449 (1912).
[7] H. A. Bethe, Ann. Physik **5**, 325 (1930).

then the integration limits are 0 and $\pi$, and expression (15) assumes an extremely simple form:

$$\sigma_{\Delta E}=\frac{\sigma_0}{(\Delta E)^2}\frac{1}{E_2}\frac{m_2}{m_1}\left[\operatorname{sgn}(-\Delta E)+\frac{4}{3}\frac{E_1}{|\Delta E|}\right] \quad \text{for} \quad v_2\geq v_1$$

$$=\frac{\sigma_0}{(\Delta E)^2}\frac{1}{E_2}\frac{v_2}{v_1}\left[\operatorname{sgn}(\Delta E)+\frac{4}{3}\frac{E_2}{|\Delta E|}\right] \quad \text{for} \quad v_2\leq v_1. \tag{17}$$

If

$$|\Delta E|\geq \Delta E_{\theta=0} \quad \text{and} \quad |\Delta E|\geq \Delta E_{\theta=\pi}, \tag{18}$$

then the integration range extends from $\theta_1$ to $\theta_2$, and we obtain accordingly:

$$\sigma_{\Delta E}=\frac{\sigma_0}{\Delta E^2}\frac{1}{E_2}\frac{m_2}{m_1}\left(1-\frac{\Delta E}{E_1}\right)^{1/2}\left[\tfrac{1}{3}\operatorname{sgn}(-\Delta E)+\frac{4}{3}\frac{E_1}{|\Delta E|}\right] \quad \text{for} \quad \Delta E\geq -\frac{\mu}{2}(v_2^2-v_1^2)$$

$$=\frac{\sigma_0}{\Delta E^2}\frac{1}{E_2}\frac{v_2}{v_1}\left(1+\frac{\Delta E}{E_2}\right)^{1/2}\left[\tfrac{1}{3}\operatorname{sgn}(\Delta E)+\frac{4}{3}\frac{E_2}{|\Delta E|}\right] \quad \text{for} \quad \Delta E\leq -\frac{\mu}{2}(v_2^2-v_1^2). \tag{19}$$

If

$$|\Delta E|\geq \Delta E_{\theta=0} \quad \text{and} \quad |\Delta E|\leq \Delta E_{\theta=\pi}, \tag{20}$$

then one integration limit is constant, 0 or $\pi$, while the other is variable, and therefore relation (15) cannot be reduced to a simpler form.

(ii) Cross section determining the dynamic friction of a particle moving in a collection of isotropically distributed field particles

$$S_D=(\sigma_0/E_1)g_D^{\text{ex}}, \tag{21}$$

$$g_D^{\text{ex}}=\left(1+\frac{m_1}{m_2}\right)\left(\frac{v_1}{v_2}\right)^2\frac{1}{2d}\left\{\frac{t-sr}{\sqrt{2}}\left[\begin{matrix}\arctan[(2x)^{1/2}/1-x] & (\text{if } x\leq 1)\\ \pi-\arctan[(2x)^{1/2}/x-1] & (\text{if } x\geq 1)\end{matrix}\right]\right.$$
$$\left.+\frac{t+sr}{\sqrt{2}}\ln\frac{x+(2x)^{1/2}+1}{(1+x^2)^{1/2}}+\frac{1}{2}\left(\frac{sr}{\sqrt{x}}+t\sqrt{x}\right)\ln(1+x^2)-2t\sqrt{x}\right\}\Bigg|_{x=r^2}^{x=s^2}, \tag{22}$$

where the following notation has been introduced:

$$r=\left(\frac{D\mu}{q_1q_2}\right)^{1/2}v_1\left(1-\frac{v_2}{v_1}\right)=d\left(1-\frac{v_2}{v_1}\right),\quad s=\left(\frac{D\mu}{q_1q_2}\right)^{1/2}v_1\left(1+\frac{v_2}{v_1}\right)=d\left(1+\frac{v_2}{v_1}\right),\quad t=\frac{m_2-m_1}{m_2+m_1}. \tag{23}$$

For the special case of heavy particles slowed down by light ones ($m_2\gg m_1$, $t=1$ and $\mu=m_1$), we obtain

$$S_D=-(4\pi(q_1q_2)^2/m_1v_2^2)G(d,v_2/v_1), \tag{24}$$

and now

$$G(d;v_2/v_1)=\frac{1}{4d}\left\{\frac{1-sr}{\sqrt{2}}\left[\begin{matrix}\arctan[(2x)^{1/2}/1-x] & (\text{if } x\leq 1)\\ \pi-\arctan[(2x)^{1/2}/x-1] & (\text{if } x\geq 1)\end{matrix}\right]\right.$$
$$\left.+\frac{1+sr}{\sqrt{2}}\ln\frac{x+(2x)^{1/2}+1}{(1+x^2)^{1/2}}+\frac{1}{2}\left(\frac{sr}{\sqrt{x}}+\sqrt{x}\right)\ln(1+x^2)-2\sqrt{x}\right\}\Bigg|_{x=r^2}^{x=s^2}. \tag{25}$$

The plots of the functions $g_\sigma^{\text{ex}}$, $g_D^{\text{ex}}$ as well as of the functions $g_Q^{\text{ex}}$ and $g_S^{\text{ex}}$, determined by the relations

$$Q=(\sigma_0/U^2)g_Q^{\text{ex}}, \tag{26}$$

$$S=(\sigma_0/U)g_S^{\text{ex}}, \tag{27}$$

[which have been obtained by numerical computations of (7) and (8) for $f(\theta)=\frac{1}{2}\sin\theta$ in the two essential cases $m_1=m_2$ and $m_1\ll m_2$] are given in Figs. 2–8.

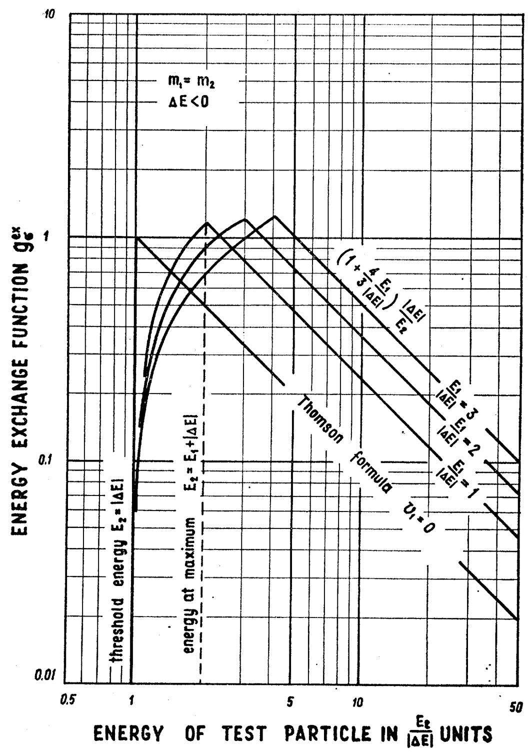


FIG. 2. Graph of the function which determines the probability of the collision in which the test particle of energy $E_2$ and mass $m_2$ loses the amount of energy $\Delta E$ if it collides with a field particle of the same mass and energy $E_1$.

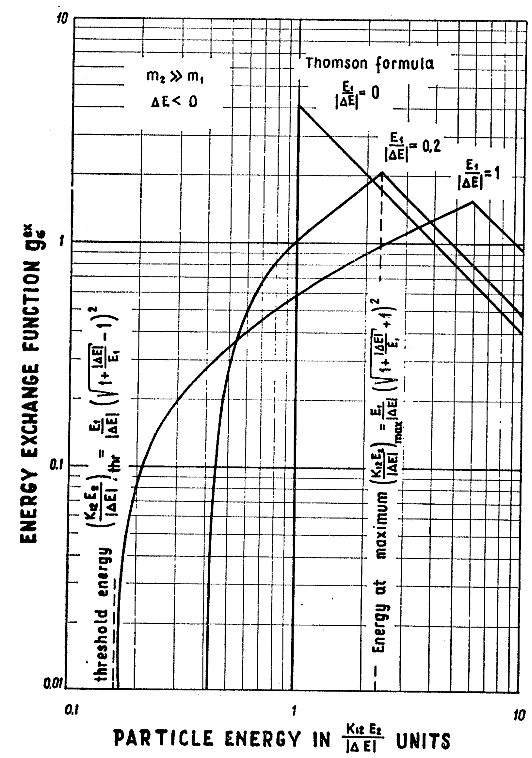


FIG. 4. Energy-exchange function $g_\sigma^{ex}$ for the loss of energy in the case of heavy test particles.

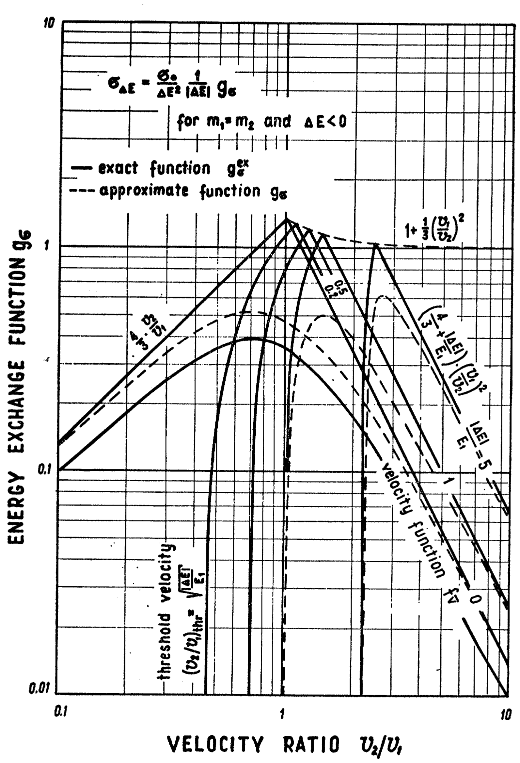


FIG. 3. Exact energy-exchange function $g_\sigma^{ex}$ and its approximation $g_\sigma$ for the loss of energy in the case of colliding particles of equal masses.

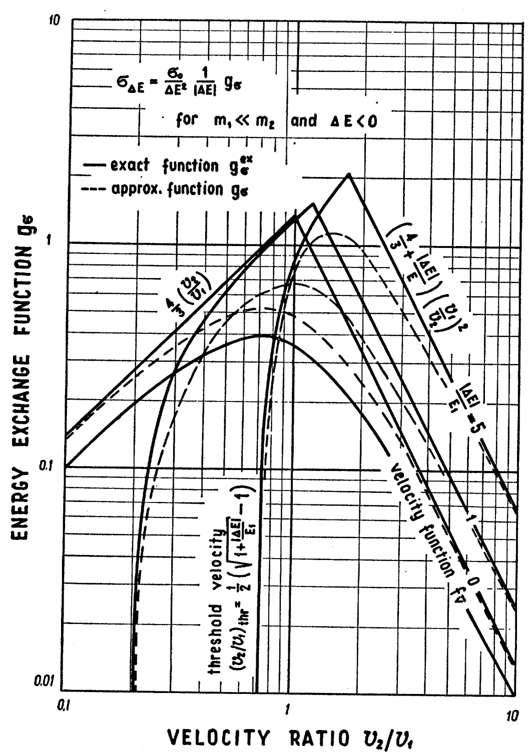


FIG. 5. Exact and approximate values of the energy-exchange function for the loss of energy versus the ratio of the velocities of colliding particles.

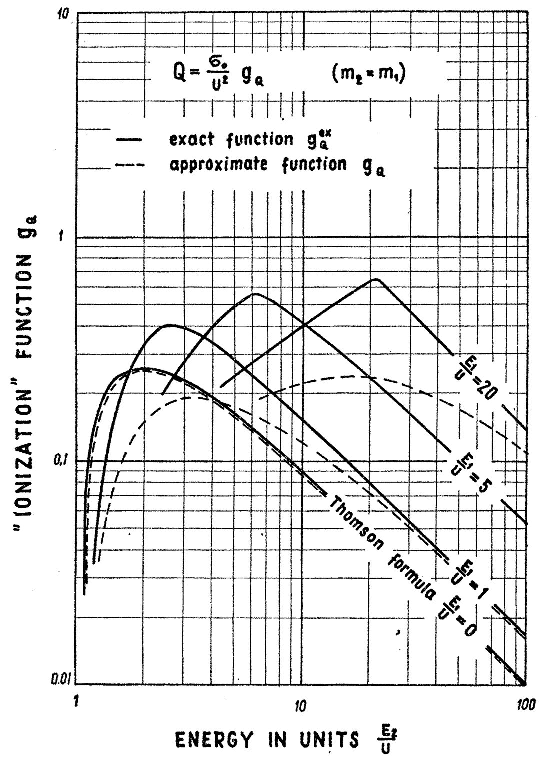


FIG. 6. Graph of the function which determines the form of the cross section for the collision in which the test particle loses an amount of energy greater than a fixed value $U$ up to the greatest possible loss $\Delta E_{max}$.

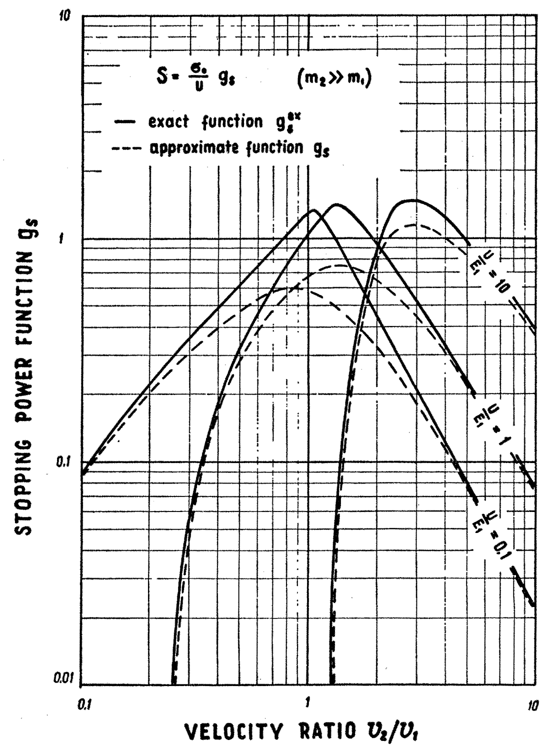


FIG. 7. Graph of the function which describes the dynamic friction of the field particles if the collisions with a loss of energy greater than $U$ are taken into account (the slowing down due to "ionizing collisions").

For $m_1 = m_2$ we can give the analytical form of the cross sections considered. Then we have, accordingly,

$$\sigma_{\Delta E} = \frac{\sigma_0}{(\Delta E)^2}\left(\frac{v_1}{v_2}\right)^2\left(\frac{4}{3}+\frac{|\Delta E|}{E_1}\right) \quad \text{if} \quad 0 \le -\frac{\Delta E}{E_2} \le 1-\frac{E_1}{E_2}$$

$$= \frac{\sigma_0}{(\Delta E)^2}\left(\frac{v_2}{v_1}\right)^2\left(\frac{4}{3}-\frac{1}{3}\frac{|\Delta E|}{E_2}\right)\left(1-\frac{|\Delta E|}{E_2}\right)^{1/2} \quad \text{if} \quad 1-\frac{E_1}{E_2} \le -\frac{\Delta E}{E_2} \le 1, \tag{28}$$

$$Q = \frac{\sigma_0}{U^2}\times\frac{2}{3}\frac{v_2}{v_1}\left(1-\frac{U}{E_2}\right)^{3/2} \quad \text{if} \quad E_2 \le E_1+U$$

$$= \frac{\sigma_0}{U^2}\left(\frac{v_1}{v_2}\right)^2\left[\frac{2}{3}+\frac{U}{E_1}\left(1-\frac{U}{E_2-E_1}\right)\right] \quad \text{if} \quad E_2 \ge E_1+U, \tag{29}$$

$$S = \frac{\sigma_0}{U}\left(\frac{v_2}{v_1}\right)^2\left[\frac{4}{3}\left(1+\frac{1}{2}\frac{U}{E_2}\right)\left(1-\frac{U}{E_2}\right)^{1/2}-\frac{U}{E_2}\ln\frac{1+(1-U/E_2)^{1/2}}{1-(1-U/E_2)^{1/2}}\right] \quad \text{if} \quad E_2 \le U+E_1$$

$$= \frac{\sigma_0}{U}\left(\frac{v_1}{v_2}\right)^2\left[\frac{4}{3}\left(1-\frac{U}{E_2-E_1}\right)+\frac{U}{E_1}\ln\frac{E_2-E_1}{U}\right.$$

$$\left. +\frac{U}{E_1}\left(\frac{E_2}{E_1}\right)^{1/2}\left(\frac{4}{3}\frac{E_2}{E_2-E_1}+\frac{2}{3}-\left(\frac{E_2}{E_1}\right)^{1/2}\ln\frac{1+(E_1/E_2)^{1/2}}{1-(E_1/E_2)^{1/2}}\right)\right] \quad \text{if} \quad E_2 \ge U+E_1. \tag{30}$$

## V. APPROXIMATIONS OF DERIVED CROSS SECTIONS

Since we have in view the application of the obtained relations to concrete calculations and to the interpretation of experimental results, our interest lies not only in exact expressions (which do not always have a physical sense, since they concern the abstract collision process of two isolated bodies), but also in the simplest possible reasonably exact approximations of the relations obtained. Thus, in the relations (6), (7), and (8) determining $\sigma_{\Delta E}$, $Q$, $S$, we approximate the function $f_V$ by the first two terms of its series expansion and neglect the dependence of $\Delta E^-$ on the angle $\theta$ by putting $\Delta E^-(\theta)=\Delta E^-_{\max}$. Also, we average the integrand over the angle $\theta$ in the range 0-$\pi$, using only the part of the integration range over $\theta$ in which (for $\Delta E\leq 0$) $\theta_{1,2}$ can be approximated by (see Fig. 9)

$$\cos\theta_{1,2}\simeq\pm\left(1-\frac{|\Delta E^-|}{E_2}\right)^{E_1/(E_1+|\Delta E^-|)} \qquad \text{for} \quad m_1=m_2, \tag{31}$$

$$\begin{aligned}&\cos\theta_1\simeq 1-2(|\Delta E^-|/|\Delta E^-_{\max}|)^{1+v_2^2/v_1^2},\\ &\cos\theta_2\simeq -1, \qquad \text{for} \quad m_1\ll m_2.\end{aligned} \tag{32}$$

Then we obtain

$$\begin{aligned}\sigma_{\Delta E}\simeq\frac{\sigma_0}{\Delta E^2}\frac{1}{|\Delta E^-|}f_V^{(0)}&\times\left[\frac{|\Delta E^-|}{E_1}\left(1-\frac{E_1}{E_2}\right)+\frac{4}{3}\right]\left(1-\frac{|\Delta E^-|}{E_2}\right)^{E_1/(E_1+|\Delta E^-|)} \quad \text{for} \quad m_1=m_2\\ &\times\left[\frac{|\Delta E^-|}{E_1}\frac{v_2^2}{v_2^2+v_1^2}+\frac{4}{3}\right]\left[1-\left(\frac{|\Delta E^-|}{|\Delta E^-_{\max}|}\right)^{1+v_2^2/v_1^2}\right] \quad \text{for} \quad m_1\ll m_2,\end{aligned} \tag{33}$$

$$\begin{aligned}Q\simeq\frac{\sigma_0}{U^2}f_V^{(0)}&\times\left[\frac{U}{E_1}\left(1-\frac{E_1}{E_2}\right)+\frac{2}{3}\left(1+\frac{U}{E_2}\right)\right]\left(1-\frac{U}{E_2}\right)^{1+E_1/(E_1+U)} \quad \text{for} \quad m_1=m_2\\ &\times\left[\frac{U}{E_1}\frac{v_2^2}{v_2^2+v_1^2}+\frac{2}{3}\left(1+\frac{U}{|\Delta E^-_{\max}|}\right)\right]\left(1-\frac{U}{|\Delta E^-_{\max}|}\right)\left[1-\left(\frac{U}{|\Delta E^-_{\max}|}\right)^{1+v_2^2/v_1^2}\right] \quad \text{for} \quad m_1\ll m_2,\end{aligned} \tag{34}$$

$$\begin{aligned}S\simeq\frac{\sigma_0}{U}f_V^{(0)}&\times\left[\frac{U}{E_1}\left(1-\frac{E_1}{E_2}\right)\ln\frac{E_2}{U}+\frac{4}{3}\left(1-\frac{U}{E_2}\right)\right]\left(1-\frac{U}{E_2}\right)^{E_1/(E_1+U)} \quad \text{for} \quad m_1=m_2\\ &\times\left[\frac{U}{E_1}\frac{v_2^2}{v_2^2+v_1^2}\ln\frac{|\Delta E^-_{\max}|}{U}+\frac{4}{3}\left(1-\frac{U}{|\Delta E^-_{\max}|}\right)\right]\left[1-\left(\frac{U}{|\Delta E^-_{\max}|}\right)^{1+v_2^2/v_1^2}\right] \quad \text{for} \quad m_1\ll m_2,\end{aligned} \tag{35}$$

where

$$f_V^{(0)}=f_V|_{\theta=\pi/2}=\left(\frac{v_1}{v_2}\right)^2\left(\frac{v_2^2}{v_2^2+v_1^2}\right) \tag{36}$$

and

$$|\Delta E^-_{\max}|=K_{12}E_2\left(1+\frac{v_1}{v_2}\right)=4E_1\left(\frac{v_2}{v_1}\right)^2\left(1+\frac{v_1}{v_2}\right)=E_1\alpha \tag{37}$$

is the maximum energy which the field particle of velocity $v_1$ and energy $E_1$ can gain in collision with the test particle of velocity $v_2$. For rough calculations as well as for the qualitative interpretation of Coulomb collisions, we can make use of still more simplified formulas. Thus, putting in the relations $v_1=0$ everywhere except for the Coulomb velocity function $f_V$ and the value determining $|\Delta E^-_{\max}|$, we obtain a set of relations:

$$\sigma_{\Delta E}=(\sigma_0/\Delta E^2)(1/E_1)f_V\,, \tag{38}$$

$$Q\simeq(\sigma_0/U)(1/E_1)f_V\left(1-\frac{U}{|\Delta E^-_{\max}|}\right), \tag{39}$$

$$S\simeq(\sigma_0/E_1)f_V\ln(|\Delta E^-_{\max}|/U)\,, \tag{40}$$

$$S_D\simeq(\sigma_0/E_1)f_V\ln[1+(\mu V^2D/q_1q_2)^2]\,, \tag{41}$$

which can be successfully used for the interpretation of the basic Coulomb collision phenomena.

Calculating the ratio $S/Q$, we obtain the mean loss of energy of the test particle per collision:

$$|\langle\Delta E^-\rangle|=\frac{S}{Q}=U\frac{\dfrac{4}{3}+\dfrac{U}{E_1}\dfrac{v_2^2}{v_2^2+v_1^2}\dfrac{\ln(|\Delta E^-_{\max}|/U)}{1-U|\Delta E^-_{\max}|}}{\dfrac{2}{3}\left(1+\dfrac{U}{|\Delta E^-_{\max}|}\right)+\dfrac{U}{E_1}\dfrac{v_2^2}{v_2^2+v_1^2}}, \tag{42}$$

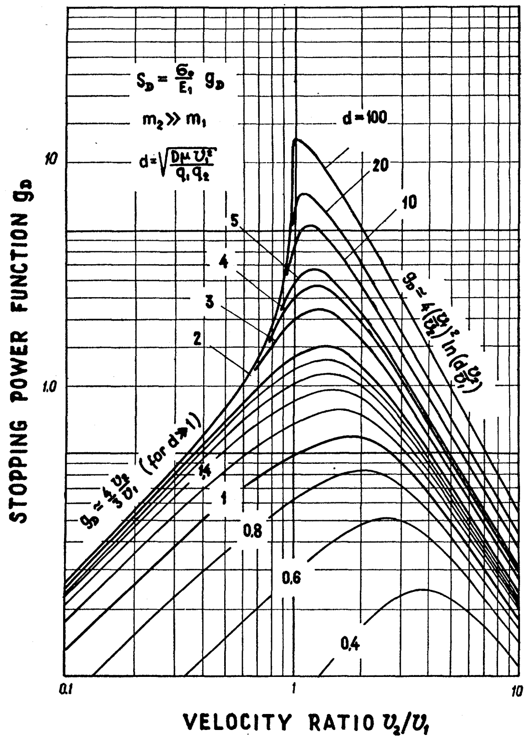


FIG. 8. Stopping-power function for heavy particles if slowed down by light ones when the collisions with loss as well as those with gain of energy are taken into account.

whence we have in the limiting cases:

(a) in the vicinity of threshold ($|\Delta E^-_{\max}| \to U$)

$$|\langle \Delta E^- \rangle| \to U; \tag{43a}$$

(b) for large velocities of test particles

$$|\langle \Delta E^- \rangle| \to U \frac{\frac{4}{3}(E_1/U)+\ln(K_{12}E_2/U)}{\frac{2}{3}(E_1/U)+1}. \tag{43b}$$

In atomic collisions $|\langle \Delta E^- \rangle|$ is simply related to the mean energy of $\delta$ electrons.

## VI. CROSS SECTIONS AVERAGED OVER THE CONTINUOUS VELOCITY DISTRIBUTION OF FIELD PARTICLES

Considering that in the majority of physical processes we have to do with a continuous velocity distribution of field particles, it is necessary to average the obtained relations over $v_1$:

$$\langle \sigma_{\Delta E} \rangle_{\rm av} = \int_0^\infty \sigma_{\Delta E}(v_1) f(v_1) dv_1,$$
$$\langle Q \rangle_{\rm av} = \int_0^\infty Q(v_1) f(v_1) dv_1, \tag{44}$$
$$\langle S \rangle_{\rm av} = \int_0^\infty S(v_1) f(v_1) dv_1.$$

Provided the maximum of the velocity distribution function $f(v_1)$ is sufficiently sharp, so that the most probable velocity differs at most slightly from the mean velocity $\bar{v}_1$, we can operate with the previously derived relations, putting into them simply $v_1 = \bar{v}_1$ and $E_1 = \frac{1}{2} m_1 \bar{v}_1^2$. However, we must keep in mind that in the case of $m_2 \gg m_1$ the threshold for inelastic process disappears in averaging over the continuous velocity distribution of field particles. This threshold, being determined by the field particles of greatest velocity, tends to zero if the velocity of the most rapid particles approaches infinity.

Nevertheless, at the threshold with respect to the mean velocity $\bar{v}_1$, which we will call hereafter the apparent threshold, the change in dependence on energy can be observed.

Below the apparent threshold, the plot of the averaged cross section is actually determined by the form of the high-energy part of the function $f(v_1)$. The cross section decreases monotonically to zero according to the decrease in the number of high-velocity field particles.

Assuming the distribution function to be of the form

$$f(v_1) = \frac{1}{(n-2)!} \left( \frac{v_1^0}{v_1} \right)^n e^{-v_1^0/v_1}, \tag{45}$$

which for region $v_1 \gg v_1^0$ may be approximated by

$$f(v_1) \simeq 1/(n-2)!(v_1^0/v_1)^n, \tag{46}$$

the energy-exchange cross section below the apparent threshold can be calculated quite easily.

Since for the region considered, all terms in (33), (34), and (35) with $v_2/v_1$ can be neglected compared to unity, we can write:

$$\langle \sigma_{\Delta E} \rangle_{\rm av}^{\rm thr} = \frac{\sigma_0}{\Delta E^3} \frac{4}{3} \int_{v_1^{\min}/v_1^0}^\infty \frac{v_2}{v_1} \left( 1 - \frac{\Delta E}{K_{12}E_2} \frac{v_2}{v_1} \right) \frac{1}{(n-2)!} \left( \frac{v_1^0}{v_1} \right)^n d\left( \frac{v_1}{v_1^0} \right),$$

$$\langle Q \rangle_{\rm av}^{\rm thr} = \frac{\sigma_0}{U^2} \frac{2}{3} \int_{v_1^{\min}/v_1^0}^\infty \frac{v_2}{v_1} \left( 1 + \frac{U}{K_{12}E_2} \frac{v_2}{v_1} \right) \left( 1 - \frac{U}{K_{12}E_2} \frac{v_2}{v_1} \right) \frac{1}{(n-2)!} \left( \frac{v_1^0}{v_1} \right)^n d\left( \frac{v_1}{v_1^0} \right),$$

$$\langle S \rangle_{\rm av}^{\rm thr} \simeq \frac{\sigma_0}{U} \frac{4}{3} \int_{v_1^{\min}/v_1^0}^\infty \frac{v_2}{v_1} \left( 1 - \frac{U}{K_{12}E_2} \frac{v_2}{v_1} \right)^2 \frac{1}{(n-2)!} \left( \frac{v_1^0}{v_1} \right)^n d\left( \frac{v_1}{v_1^0} \right),$$

$$|\Delta E^-_{\max}| \simeq K_{12}E_2 \frac{v_1}{v_2}.$$

Performing the integration over $v_1$, the lower limit of integration is obviously $v_1^{\min}=(U/K_{12}E_2)v_2$, and substituting $\frac{1}{2}m_1(v_1^0)^2=E_1^0$ we obtain successively

$$\langle\sigma_{\Delta E^-}\rangle_{\rm av}{}^{\rm thr}\simeq\frac{\sigma_0}{\Delta E^2}\frac{1}{|\Delta E^-|}\frac{4}{3}\left(\frac{v_2}{v_1^0}\right)^{n+1}\times\left(4\frac{E_1^0}{|\Delta E^-|}\right)^n\frac{1}{(n-2)!}\left(\frac{1}{n}-\frac{1}{n+1}\right), \quad (47)$$

$$\langle Q\rangle_{\rm av}{}^{\rm thr}\simeq\frac{\sigma_0}{U^2}\frac{2}{3}\left(\frac{v_2}{v_1^0}\right)^{n+1}\left(4\frac{E_1^0}{U}\right)^n\frac{1}{(n-2)!}\times\left(\frac{1}{n}-\frac{1}{n+1}-\frac{1}{n+2}-\frac{1}{n+3}\right), \quad (48)$$

$$\langle S\rangle_{\rm av}{}^{\rm thr}\simeq\frac{\sigma_0}{U}\frac{4}{3}\left(\frac{v_2}{v_1^0}\right)^{n+1}\left(4\frac{E_1^0}{U}\right)\frac{1}{(n-2)!}\times\left(\frac{1}{n}-\frac{2}{n+1}+\frac{1}{n+2}\right). \quad (49)$$

For $n=3$, which, as we shall see later, is the most important case for atomic collisions,

$$\bar{v}_1=\int v_1 f(v_1)dv_1=v_1^0.$$

Thus (47), (48), and (49) result in:

$$\langle\sigma_{\Delta E^-}\rangle_{\rm av}{}^{\rm thr}=\frac{\sigma_0}{\Delta E^2}\frac{1}{|\Delta E^-|}\frac{64}{9}\left(\frac{v_2}{\bar{v}_1}\right)^4\left(\frac{\mathcal{E}_1}{|\Delta E^-|}\right)^3, \quad (50)$$

$$\langle Q\rangle_{\rm av}{}^{\rm thr}=\frac{\sigma}{U^2}\frac{32}{15}\left(\frac{v_2}{\bar{v}_1}\right)^4\left(\frac{\mathcal{E}_1}{U}\right)^3, \quad (51)$$

$$\langle S\rangle_{\rm av}{}^{\rm thr}=\frac{\sigma_0}{U}\frac{128}{45}\left(\frac{v_2}{\bar{v}_1}\right)^4\left(\frac{\mathcal{E}_1}{U}\right)^3. \quad (52)$$

In the above, the symbol $\mathcal{E}_1$ has been used for $\frac{1}{2}m_1\bar{v}_1^2$.

If the distribution function $f(v_1)$ does not decrease rapidly enough with the decrease of $v_1$, then, on averaging over field-particle velocity distribution, some elements appear which substantially influence the averaged cross sections $\sigma$, $Q$, $S$. At $n=3$ a logarithmic term increasing with the ratio $v_2/\bar{v}_1$ appears, which cannot be neglected. This is bound up with the fact that at $n=3$ the mean kinetic energy of the field particle is logarithmically divergent. Considering that for $v_2\gg v_1$,

$$\sigma, Q, S\propto[+(v_1/v_2)^2]1/v_2^2$$

and for $v_2\ll v_1$,

$$\sigma, Q, S\propto[+(v_1/v_2)^2](v_2/v_1)^3,$$

then, averaging these cross sections over $f|v_1|$, at $n=3$ we obtain:

$$\sigma, Q, S\propto[+(\bar{v}_1/v_2)^2\ln(\alpha(v_2/\bar{v}_1))^k],$$

where $\alpha$ and $k$ are of the order of 1 and depend on the particular form of the function $f(v_1)$.

Without going into details, and taking into consideration only the first-order correction arising from the logarithmic term, we can give the approximate forms of the averaged cross sections for $n=3$:

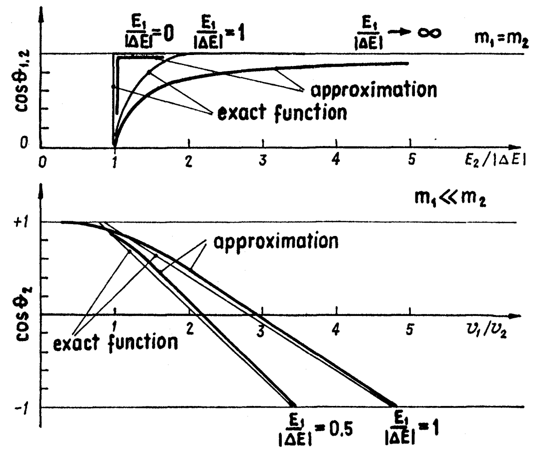


FIG. 9. Comparison of the exact and the approximate limits for the angle $\theta$ where the latter have been used for the derivation of approximate energy-exchange formulas.

$$\langle\sigma_{\Delta E}\rangle_{\rm av}{}^{\ln}\simeq\frac{\sigma_0}{\Delta E^2}\frac{1}{|\Delta E|}f_V{}^{(0)}\left[\frac{|\Delta E|}{\mathcal{E}_1}\frac{v_2^2}{v_2^2+\bar{v}_1^2}+\frac{4}{3}\ln\left(2.7+\frac{v_2}{\bar{v}_1}\right)\right]\left[1-\left(\frac{|\Delta E|}{|\Delta E^-{}_{\max}|}\right)^{1+v_2^2/\bar{v}_1^2}\right], \quad (53)$$

$$\langle Q\rangle_{\rm av}{}^{\ln}=\frac{\sigma_0}{U^2}f_V{}^{(0)}\left[\frac{U}{\mathcal{E}_1}\frac{v_2^2}{v_2^2+\bar{v}_1^2}+\frac{2}{3}\left(1+\frac{U}{|\Delta E^-{}_{\max}|}\right)\ln\left(2.7+\frac{v_2}{\bar{v}_1}\right)\right]\left(1-\frac{U}{|\Delta E^-{}_{\max}|}\right)\left[1-\left(\frac{U}{|\Delta E^-{}_{\max}|}\right)^{1+v_2^2/\bar{v}_1^2}\right], \quad (54)$$

$$\langle S\rangle_{\rm av}{}^{\ln}=\frac{\sigma_0}{U}f_V{}^{(0)}\left[\frac{U}{\mathcal{E}_1}\frac{v_2^2}{v_2^2+\bar{v}_1^2}\ln\frac{|\Delta E^-{}_{\max}|}{U}+\frac{4}{3}\left(1-\frac{U}{|\Delta E^-{}_{\max}|}\right)\ln\left(2.7+\frac{v_2}{\bar{v}_1}\right)\right]\left[1-\left(\frac{U}{|\Delta E^-{}_{\max}|}\right)^{1+v_2^2/\bar{v}_1^2}\right]. \quad (55)$$

For $m_1=m_2$, in view of (28), the presence of a term proportional to $E_1$ (for $v_2\to\infty$) is not determined by $v_2>v_1$, but by a stronger energy condition:

$$E_2\geq E_1-|\Delta E^-|.$$

As a result, the presence of the logarithmic term at $n=3$ is connected with the particles of energy $E_1 \leq E_2+|\Delta E^-|$, this leading to the relation

$$\langle\sigma_{\Delta E}\rangle_{\text{av}}^{\ln}=\frac{\sigma_0}{\Delta E^2}\frac{1}{|\Delta E|}f_V^{(0)}\left[\frac{|\Delta E|}{\mathcal{E}_1}\left(1-\frac{\mathcal{E}_1}{E_2}\right)+\frac{4}{3}\ln\left(2.7+\left(\frac{E_2-|\Delta E|}{\mathcal{E}_1}\right)^{1/2}\right)\right]\left(1-\frac{|\Delta E|}{E_2}\right)^{\mathcal{E}_1/(\mathcal{E}_1+|\Delta E|)}, \tag{56}$$

as well as to

$$\langle Q\rangle_{\text{av}}^{\ln}=\frac{\sigma_0}{U^2}f_V^{(0)}\left[\frac{U}{\mathcal{E}_1}+\frac{2}{3}\left(1-\frac{U}{2E_2}\right)\ln\left(2.7+\left(\frac{E_2-U}{\mathcal{E}_1}\right)^{1/2}\right)\right]\left(1-\frac{U}{E_2}\right)^{1+\mathcal{E}_1/(\mathcal{E}_1+U)}, \tag{57}$$

and similarly to

$$\langle S\rangle_{\text{av}}^{\ln}=\frac{\sigma_0}{U}f_V^{(0)}\left[\frac{U}{\mathcal{E}_1}\left(1-\frac{\mathcal{E}_1}{E_2}\right)\ln\frac{E_2}{U}+\frac{4}{3}\left(1-\frac{U}{E_2}\right)\ln\left(2.7+\left(\frac{E_2-U}{\mathcal{E}_1}\right)^{1/2}\right)\right]\left(1-\frac{U}{E_2}\right)^{\mathcal{E}_1/(\mathcal{E}_1+U)}. \tag{58}$$

## VII. CROSS SECTIONS $\sigma$, $Q$, $S$ IN THE RELATIVISTIC ENERGY RANGE

As has previously been shown by the author,[3] the function $Q$, and so the functions $\sigma$ and $S$, describe the inelastic collision process correctly in the range of relativistic energies, provided the velocities of the colliding particles are derived from the relativistic relations. Moreover, in accordance with what has been stated above on the general dependence of the energy-exchange cross sections on the function $f_V$, it will be enough to derive the relativistic function $f_V$.

Considering that in the relativistic case $V=(v_2^2+v_1^2(1-v_2^2/c^2))^{1/2}$ (relativistic composition of two perpendicular velocity vectors[8]) $v_1=c\{1-((1+E_1/m_{01}c^2)^{-1})^2\}^{1/2}$ and similarly for $v_2$, we have

$$f_V^{(0)\,\text{rel}}=\frac{\kappa_1}{\kappa_2}\frac{2+\kappa_1}{2+\kappa_2}\left(\frac{1+\kappa_2}{1+\kappa_1}\right)^2\left[\frac{(1+\kappa_2)^2}{(1+\kappa_2)^2+(\kappa_1/\kappa_2)((2+\kappa_1)/(2+\kappa_2))((1+\kappa_2)/(1+\kappa_1))^2}\right]^{3/2}, \tag{59}$$

where we have put

$$\kappa_1=E_1/m_{01}c^2 \quad\text{and}\quad \kappa_2=E_2/m_{02}c^2, \tag{60}$$

where $m_{01}$ and $m_{02}$ are the rest masses of the colliding particles. The graph of the function $f_V^{(0)\,\text{rel}}$ is presented in Fig. 10.

At $\kappa_1\to 0$ and $\kappa_2\to 0$ we obtain a nonrelativistic function $f_V^{(0)}$. At $\kappa_1=1$, the function $f_V^{(0)\,\text{rel}}$ is deprived of the maximum, while at $\kappa_1\to\infty$ it tends to the unit function analogically as in the case of the collision of perfectly rigid spheres.

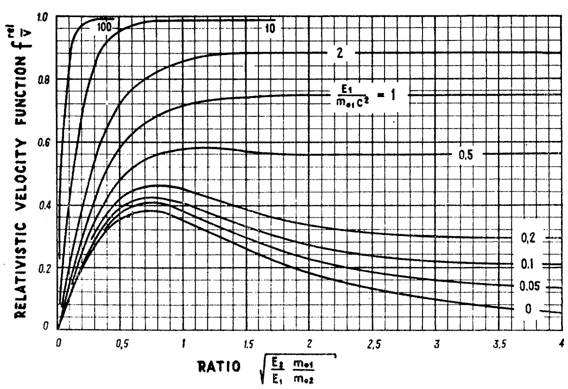


FIG. 10. Coulomb velocity function $f_V^{(0)}$ in the relativistic energy range where the relativistic formulas for the calculation of $v_1$, $v_2$, and $V$ have been used.

Thus, in the range of the relativistic velocities of the colliding particles, the cross sections $\sigma$, $Q$, $S$ take the form

$$\begin{aligned}\sigma_{\text{rel}}&\simeq\sigma\cdot f_V^{(0)\,\text{rel}}/f_V^{(0)},\\ Q_{\text{rel}}&\simeq Q\cdot f_V^{(0)\,\text{rel}}/f_V^{(0)},\\ S_{\text{rel}}&\simeq S\cdot f_V^{(0)\text{rel}}/f_V^{(0)}.\end{aligned} \tag{61}$$

The extension of the application of cross section formulas so made into the region of relativistic energies, although it describes correctly a number of physical phenomena, is a rough approximation, and it cannot constitute a basis for significant conclusions. Therefore, in order to develop the collision theory, an attempt ought to be made at deriving analogous relations based on the relativistic laws of conservation of energy and momentum.

## VIII. SCATTERING CROSS SECTIONS FOR COULOMB COLLISIONS

In a great number of collision problems we are interested not only in the cross sections for the energy exchange between the colliding particles, but also in their angular distribution. According to the asymmetry of the description of collisions in the laboratory system

[8] L. Landau and E. Lipshitz, *Classical Theory of Fields* (GITTL, Mockva-Leningrad, 1948), p. 23.

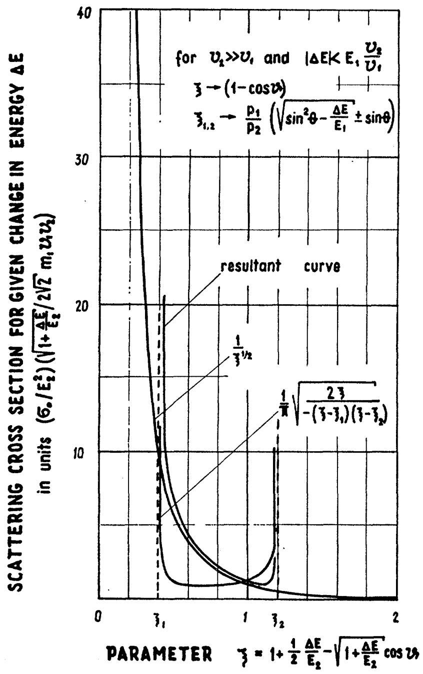


FIG. 11. Differential scattering cross section for the collisions of a given change in energy in the case of anisotropic distribution of field particles; the figure illustrates the appearance of "diffraction" maxima.

of coordinates, determined by the direction of the velocity vector of the test particle, we shall speak about the angular distribution of scattered particles and the angular distribution of recoiled particles. The term "scattered" will refer to the test particle and the term "recoiled" will refer to the field particle.

The cross section for the scattering of the test particle at the angle $\vartheta$ with the change in energy $\Delta E$ is given by relation (2).

In the case of a strongly anisotropic velocity distribution of field particles (Fig. 11), if this can be approximated by the $\delta$ function

$$f(\theta)=\delta(\theta-\theta_0),$$

we simply obtain:

$$\sigma_{\Delta E,\cos\vartheta}=\frac{\sigma_0}{E_2^2}\frac{(1+\Delta E/E_2)^{1/2}}{\xi^{3/2}}\times\frac{1}{2\sqrt{2}m_1v_1v_2}\frac{1}{\pi}\frac{1}{(W_\xi(\theta_0))^{1/2}}, \quad (62)$$

or, taking into account that $W_\xi$ can be expressed in the alternative form,

$$\sigma_{\Delta E,\cos\vartheta}=\frac{\sigma_0}{E_2^2}\frac{(1+\Delta E/E_2)^{1/2}}{\xi^2}\times\frac{1}{m_1v_1v_2}\frac{1}{2\pi}\frac{1}{((\xi-\xi_1)(\xi-\xi_2))^{1/2}}, \quad (63)$$

where $\xi_{1,2}(\theta_0,\Delta E/E_2,v_1/v_2)$ are the roots of the equation $W_\xi=0$ which are given by (I.96) and (I.97). The position of the singularities in the cross section $\sigma_{\Delta E,\cos\vartheta}$, which is independent of the law of interaction, has been carefully discussed in Paper I.

The cross section $\sigma_{\Delta E,\cos\vartheta}$ in the form written above is very useful if we discuss it with respect to $\Delta E$ or to the angle $\vartheta$. To have a clear dependence on the velocities of colliding particles we rewrite (62) in a different form:

$$\sigma_{\Delta E,\cos\vartheta}=\frac{\sigma_0}{E_2^2}\frac{(1+\Delta E/E_2)^{1/2}}{\xi^2}\frac{1}{E_2}\frac{1}{4\pi}\frac{1}{\sqrt{W_v}}, \quad (64)$$

where now

$$W_v=-\left(\frac{v_1}{v_2}\right)^2\left[\left(\xi-\frac{1}{2}\frac{\Delta E}{E_2}\right)^2-2\xi\sin^2\theta\right]+2\left(\frac{v_1}{v_2}\right)\cos\theta\left(\frac{m_2}{m_1}\xi+\frac{1}{2}\frac{\Delta E}{E_2}\right)\left(\frac{1}{2}\frac{\Delta E}{E_2}-\xi\right)-\left(\frac{m_2}{m_1}\xi+\frac{1}{2}\frac{\Delta E}{E_2}\right)^2. \quad (65)$$

If $v_1\to 0$, then obviously $(m_2/m_1)\xi+\frac{1}{2}(\Delta E/E_2)\to 0$, and there appears a unique relation between the scattering angle $\vartheta$ and the loss of energy $\Delta E$.

In the case of isotropic distribution of the field particles, (2) results:

$$\sigma_{\Delta E,\cos\vartheta}=\frac{\sigma_0}{E_2^2}\frac{(1+\Delta E/E_2)^{1/2}}{\xi^{5/2}}\frac{1}{2\sqrt{2}m_2v_1v_2} \quad \text{for} \quad \xi_1\le\xi\le\xi_2$$
$$=0 \quad \text{for} \quad \xi<\xi_1 \quad \text{or} \quad \xi<\xi_2 \quad (66)$$

with the limits for $\xi$ as given by (I.103).

Integrating $\sigma_{\Delta E,\cos\vartheta}$ over the whole range of variability of angle $\vartheta$ [see (I.104)], we obviously obtain the cross section $\sigma_{\Delta E}$; on the other hand, integrating over all possible values of $\Delta E$, we obtain the cross section for scattering at the angle $\vartheta$. Although the integration of (66) over $\Delta E$ can be performed, the analytical form of $\sigma_{\cos\vartheta}$ in the general case cannot be given because of the difficulty in solving the fourth-power equation defining the limits of the integral.

The scattering cross section $\sigma_{\cos\vartheta}$ in the explicit form can be obtained for parallel or antiparallel velocities of the colliding particles only. Inserting then in (I.80) $f(\theta)=\delta(\theta)$ or $f(\theta)=\delta(\theta-\pi)$, we have

$$\sigma_{\cos\vartheta}=\left[\frac{q_1q_2}{\mu(v_2\mp v_1)^2}\right]^2\frac{|v_2\mp v_1|}{v_2}\sum_{i=1}^{2}\int\frac{d(1/\cos^2\Psi_g)}{(-(\Delta E_i+b\cos^2\Psi_g)^2)^{1/2}}2E_2\frac{(1+\Delta E_i/E_2)^{3/2}}{|2M_1(1\mp v_1/v_2)^2\cos^2\Psi_g+\frac{1}{2}(\Delta E_i/E_2)|},\tag{67}$$

where $\Delta E_i$ is given by (I.81), and the upper sign is for the parallel velocities while the lower is for the antiparallel. Because $\sigma_{\cos\vartheta}$ must be a real quantity, the range of integration over the angle $\Psi_g$ reduces to the calculation of the residues of the poles of the function under the integral. Taking into account that the poles are determined by

$$(1/\cos^2\Psi_g)=\Delta E_i/b,$$

and the integrand has poles of first order,

$$\int\frac{xdx}{(-(\Delta E_ix+b)^2)^{1/2}}=\pi\frac{b}{\Delta E_i^2},$$

we can write

$$\sigma_{\cos\vartheta}=2\pi\left[\frac{q_1q_2}{\mu(v_2\mp v_1)^2}\right]^2\frac{|v_2\mp v_1|}{v_2}\times\sum_{i=1}^{2}\frac{bE_2}{\Delta E_i^2}\frac{(1+\Delta E_i/E_2)^{3/2}}{|1-4M_1^2(1\mp v_1/v_2)^2E_2/b|}\frac{2E_2}{|\Delta E_i|}.\tag{68}$$

Now, the condition (I.81) takes the form:

$$\cos\vartheta=(1+\Delta E_1/E_2)^{-1/2}(1+\tfrac{1}{2}(\Delta E_i/E_2)R),\tag{69}$$

where we have set

$$R=\frac{1+m_1/m_2}{1\pm p_1/p_2}.\tag{70}$$

Finally, after making some transformations, the scattering cross section in the case of parallel or antiparallel field-particle velocities is

$$\sigma_{\cos\vartheta}=2\pi\left(\frac{q_1q_2}{m_2v_2^2}\right)^2\frac{v_2}{v_1}\frac{p_1}{|p_1\pm p_2|}\frac{R}{(1-R)^2}\frac{1}{\sin^4\vartheta}\left(1-\frac{2}{R}+\frac{\cos^2\vartheta}{R^2}\right)^{-1/2}$$
$$\times\left[\left(1-\frac{1}{R}\right)\cos\vartheta+\left(1-\frac{2}{R}+\frac{\cos^2\vartheta}{R^2}\right)^{1/2}\right]\quad\text{if}\quad R\geq 2\quad\text{or}\quad R\leq 0$$
$$\times 2\left[\left(1-\frac{1}{R}\right)^2\cos^2\vartheta+\left(\left(1-\frac{2}{R}+\frac{\cos^2\vartheta}{R^2}\right)^{1/2}\right)^2\right]\quad\text{if}\quad 0\leq R\leq 2.\tag{71}$$

Taking into account that

$$1-2/R+\cos^2\vartheta/R^2\geq 0\tag{72}$$

and comparing with (69), we deduce that if $R\leq 0$ or $R\geq 2$, then the forward scattering as well as the back scattering is possible, and if $0\leq R\leq 2$ then only the forward scattering is possible. Assuming that $v_1=0$, we obtain directly from (71) the cross section for scattering of the test particle at an angle $\vartheta$ by the field particle at rest:

$$\sigma_{\cos\vartheta}=\pi\left(\frac{q_1q_2}{m_2v_2^2}\right)^2\frac{1}{\sin^4\vartheta}\frac{2}{(1-(m_2/m_1)^2\sin^2\vartheta)^{1/2}}$$
$$\times[\cos\vartheta+(1-(m_2/m_1)^2\sin^2\vartheta)^{1/2}]^2\quad\text{if}\quad m_2\leq m_1$$
$$\times 2[1+\cos^2\vartheta-(m_2/m_1)^2\sin^2\vartheta]^2\quad\text{if}\quad m_2\geq m_1\tag{73}$$

which in the case $m_1\to\infty$ (scattering by a center of force) gives us the Rutherford formula

$$\sigma_{\cos\vartheta}=\pi(q_1q_2/m_2v_2^2)^2 2(1+\cos\vartheta)^2/\sin^4\vartheta;\tag{74}$$

or in the case of equal masses of the two particles:

$$\sigma_{\cos\vartheta}=\pi(q_1q_2/m_2v_2^2)^2(8\cos\vartheta/\sin^4\vartheta).\tag{75}$$

If the mass of the test particle is greater than that of the field particle ($m_2\gg m_1$), then the differential scattering cross section is divergent not only for $\vartheta\to 0$ but also for $\vartheta\to\arcsin(m_1/m_2)$ (see Fig. 12). Nevertheless, the total scattering cross section integrated over the range $\theta_0\leq\theta\leq\arcsin(m_1/m_2)$ is finite. The presence of that divergence is obvious if we examine the angle of scattering as a function of the impact parameter $D$. In the case of field particles at rest, for the Coulomb interaction Eq. (I.110), we get

$$\cos\vartheta=\left[1-2\frac{m_1}{m_1+m_2}\frac{1}{1+(D\mu v_2^2/q_1q_2)^2}\right]\times\left[1-4\frac{m_1m_2}{(m_1+m_2)^2}\frac{1}{1+(D\mu v_2^2/q_1q_2)^2}\right]^{-1/2}.\tag{76}$$

The graph of the relation obtained is shown in Fig. 13.

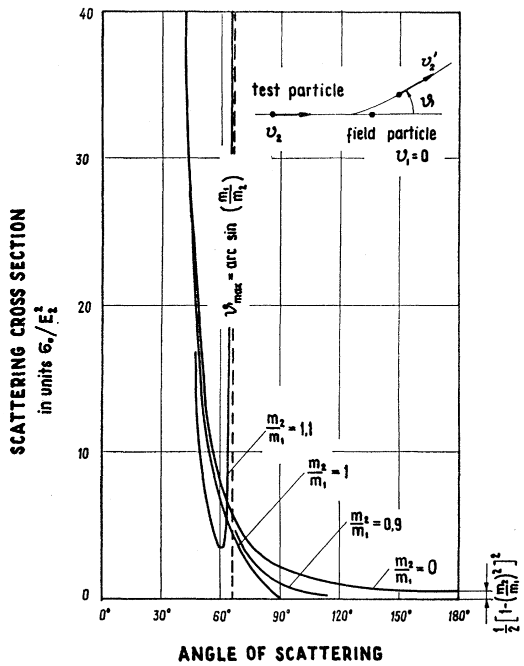


FIG. 12. Differential scattering cross section for scattering on particles at rest, for colliding particles of different masses.

We see that the divergence of the scattering cross section for $m_2>m_1$ occurs for the impact parameter at which $d\vartheta/dD=0$.

In the same case of $m_2>m_1$, the test particle can be scattered at the same angle $\vartheta$ at two different values of the impact parameter $D$. This is the reason why in (67) there appears a summation over one or two values of (i). We have a similar picture in the general case of $\sin\theta\neq 0$, as well as for forces different from the Coulomb forces if only they decrease monotonically with the distance between the interacting particles.

Expression (3) determining the scattering cross section with respect to angle $\vartheta$ as well as angle $\phi$ makes sense if there is anisotropy in the velocities of the field particles with respect to angle $\varphi$. Then, according to the relation (I.32), some values of angle $\phi$ are excluded. For $v_2\gg v_1$ and $|\Delta E^-|<E_1v_2/v_1$, according to the results in Paper I, we have $\phi\simeq\varphi$, and therefore we can write

$$\sigma_{\Delta E^-,\cos\vartheta,\phi}\simeq\sigma_{\Delta E^-,\cos\vartheta}f(\varphi)_{\varphi=\phi}\,, \qquad (77)$$

where $f(\varphi)$ is a velocity distribution function with respect to the angle $\varphi$.

## IX. SCATTERING CROSS SECTION AVERAGED OVER VELOCITY DISTRIBUTION OF FIELD PARTICLES

Similar to our treatment of the energy cross sections, we may average the scattering cross sections over the velocity distribution of the field particles. In the case of a continuous velocity distribution, according to Paper I, the possible range of scattering angle $\vartheta$ is from zero to $\pi$. From (66), assuming that the velocity distribution has the form given by formula (45) with $n=3$, we have

$$\langle\sigma_{\Delta E,\cos\vartheta}\rangle_{\rm av}=\frac{\sigma_0}{E_2{}^2}\frac{(1+\Delta E/E_2)^{1/2}}{\xi^{3/2}}\frac{1}{2\sqrt{2}\bar{v}_1v_2}\int_{x_0}^{\infty}\frac{e^{-1/x}}{x^4}dx\,, \qquad (78)$$

where $x_0=(v_1/v_2)_{\rm min}$ is determined by the condition (I.102); or explicitly

$$x_0=\left(\frac{v_2}{\bar{v}_1}\right)\left(\frac{\xi}{2}\right)^{1/2}\left(\frac{m_2}{m_1}+\frac{1}{2}\frac{\Delta E}{E_2}\frac{1}{\xi}\right). \qquad (79)$$

Having performed the integration, we have

$$\langle\sigma_{\Delta E,\cos\vartheta}\rangle_{\rm av}=\frac{\sigma_0}{E_2{}^2}\frac{(1+\Delta E/E_2)^{1/2}}{\xi^{3/2}}\frac{1}{2\sqrt{2}m_1\bar{v}_1v_2}$$

$$\times 2\left[1-e^{-1/x_0}\left(1+\frac{1}{x_0}+\frac{1}{2x_0{}^2}\right)\right]. \qquad (80)$$

The graph of the obtained $\langle\sigma_{\Delta E,\cos\vartheta}\rangle_{\rm av}$ is shown in Fig. 14. As can be easily seen from Fig. 14, the "diffraction" maximum of "inelastically" scattered particles is approximately given by

$$2\left(\frac{m_2v_2}{m_1v_1}\right)^2\left(1-\frac{1}{2}\frac{|\Delta E^-|}{E_2}-\left(1-\frac{|\Delta E^-|}{E_2}\right)^{1/2}\cos\vartheta\right)\simeq\frac{|\Delta E^-|}{E_1}-1. \qquad (81)$$

For $|\Delta E^-|/E_2\ll 1$ and $\vartheta\to 0$, we obtain

$$|\Delta E^-|\simeq E_0+(m_2/m_1)E\vartheta^2. \qquad (82)$$

Performing the integration of $\langle\sigma_{\Delta E,\cos\vartheta}\rangle_{\rm av}$ over $|\Delta E|$ in the range $U\leq|\Delta E^-|\leq|\Delta E^-{}_{\rm max}|$, we shall obtain the cross section for the scattering of the test particle at the angle $\vartheta$ if the particle loses the energy in the interval mentioned above (in atomic physics this cross section determines the angular distribution of particles scat-

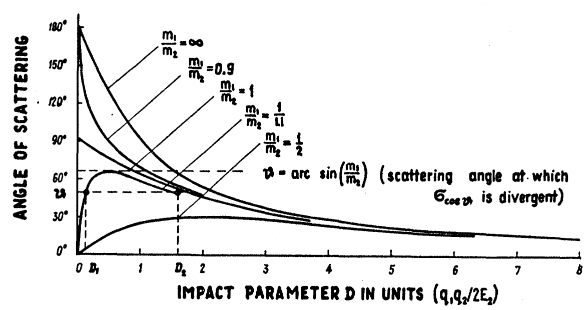


FIG. 13. Dependence of the angle of scattering on the impact parameter $D$, which illustrates the appearance of the divergence of the differential scattering cross section.

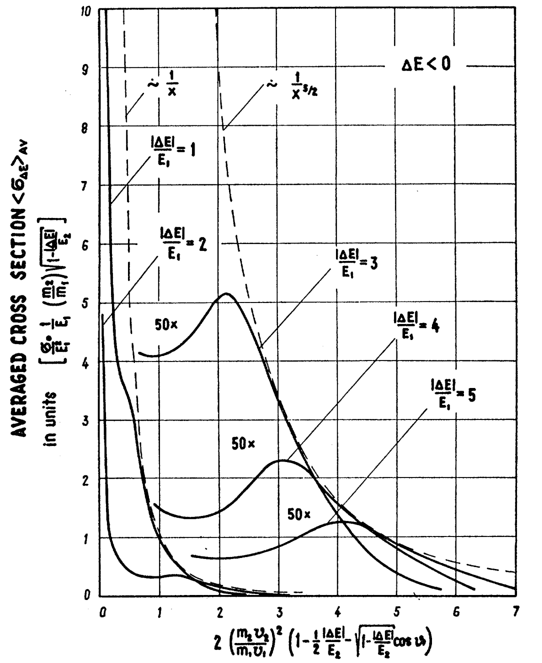


FIG. 14. Differential scattering cross section for collisions of given change in energy if averaged over the continuous velocity distribution of field particles of the form given by Eq. (46) for $n=3$.

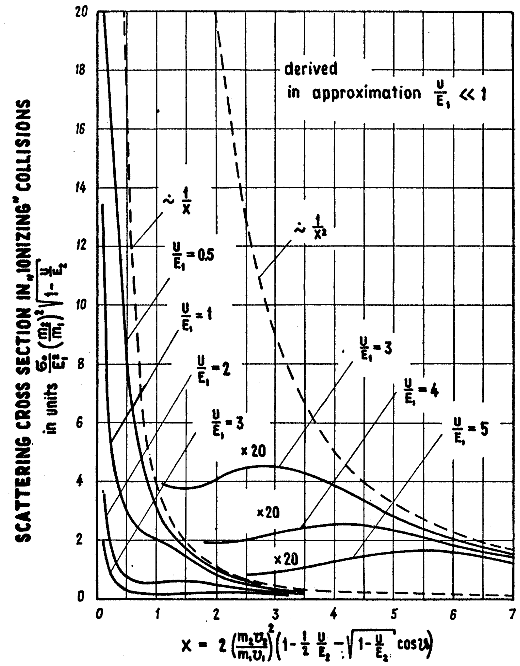


FIG. 15. Scattering cross section in "ionizing" collisions averaged over the velocity distribution of field particles. The "diffraction" maxima are distinctly visible on the curve for $U \gg E_1$.

tered in ionization collisions):

$$\langle Q_{U,\cos\vartheta}\rangle_{\rm av} = \int_{-U}^{-|\Delta E^-_{\rm max}|} \langle \sigma_{\Delta E,\cos\vartheta}\rangle_{\rm av} d(\Delta E). \quad (83)$$

The explicit form of $\langle Q_{U,\cos\vartheta}\rangle_{\rm av}$ can be obtained in the approximation $U/E_2 \ll 1$. In this case we can write

$$\langle Q_{U,\cos\vartheta}\rangle_{\rm av} \simeq \frac{\sigma_0}{E_2^2} \frac{(1-U/E_2)^{1/2}}{\xi^{5/2}(U/E_2)} \times \int_{x_0(-U)}^{x_0(|\Delta E^-_{\rm max}|)} \left[1-e^{-1/x_0}\left(1+\frac{1}{x_0}+\frac{1}{2x_0^2}\right)\right] dx_0, \quad (84)$$

where $x_0$ is given by (79). Performing the integration, we obtain

$$\langle Q_{U,\cos\vartheta}\rangle_{\rm av} = \frac{\sigma_0}{E_2^2} \frac{(1-U/E_2)^{1/2}}{\xi^2(U/E_2)} \times \{\tfrac{1}{2} \pm [x_0(1-e^{-1/x_0}) - \tfrac{1}{2}e^{-1/x_0}\}, \quad (85)$$

where the upper sign applies when $2(p_2/p_1)^2\xi \geq U/E_1$ and the lower one when $2(p_2/p_1)^2\xi < U/E_1$. The graph of the cross section obtained is shown in Fig. 15.